

Homological algebra methods in the theory of Operator Algebras II, Homological functors, derived functors, Adams spectral sequence, assembly map and BC for compact quantum groups.

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Let \mathfrak{C} be an abelian category

We call a covariant functor $F: \mathfrak{T} \rightarrow \mathfrak{C}$ *homological* if

$$F(C) \rightarrow F(A) \rightarrow F(B)$$

is exact for any distinguished triangle

$$\Sigma B \rightarrow C \rightarrow A \rightarrow B.$$

We define $F_n(A) := F(\Sigma^n A)$ for $n \in \mathbb{Z}$.

Similarly, we call a contravariant functor $F: \mathfrak{T} \rightarrow \mathfrak{C}$ *cohomological* if $F(B) \rightarrow F(A) \rightarrow F(C)$ is exact for any distinguished triangle, and we define $F^n(A) := F(\Sigma^n A)$.

We will always be in the following situation.

- 1 \mathcal{C} is some abelian category equipped with a shift $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ (\mathcal{C} is stable).
- 2 Our functors $F : \mathcal{T} \rightarrow \mathcal{C}$ are homological and stable, i. e. commute with Σ .

We will do homological algebra relative to some ideal in \mathfrak{T} which satisfies the following property.

Definition

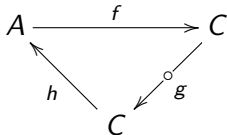
An ideal $\mathfrak{I} \subseteq \mathfrak{T}$ is called *stable* if the suspension isomorphisms $\Sigma: \mathfrak{T}(A, B) \xrightarrow{\cong} \mathfrak{T}(\Sigma A, \Sigma B)$ for $A, B \in \mathfrak{T}$ restrict to isomorphisms

$$\Sigma: \mathfrak{I}(A, B) \xrightarrow{\cong} \mathfrak{I}(\Sigma A, \Sigma B).$$

Definition

An ideal $\mathfrak{I} \subseteq \mathfrak{T}$ in a triangulated category is called *homological* if it is the kernel of a stable homological functor.

Before continuing with definitions, recall the picture of distinguished triangle in \mathfrak{T} .



Definition

Let \mathfrak{T} be a triangulated category and let $\mathcal{I} \subseteq \mathfrak{T}$ be a homological ideal. Let $f: A \rightarrow B$ be a morphism in \mathfrak{T} and embed it in an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$.

- We call f \mathcal{I} *monic* if $h \in \mathcal{I}$.
- We call f \mathcal{I} *epic* if $g \in \mathcal{I}$.
- We call f an \mathcal{I} *equivalence* if f is both \mathcal{I} monic and \mathcal{I} epic or, equivalently, if $g, h \in \mathcal{I}$.
- We call f an \mathcal{I} *phantom map* if $f \in \mathcal{I}$.

An object $A \in \mathfrak{T}$ is called \mathcal{I} *contractible* if $\text{id}_A \in \mathcal{I}(A, A)$.

An exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in \mathfrak{T} is called \mathcal{I} *exact* if $h \in \mathcal{I}$.

Consider a chain complex $C_\bullet = (C_n, d_n)$. For each $n \in \mathbb{N}$, we may embed the map d_n in an exact triangle

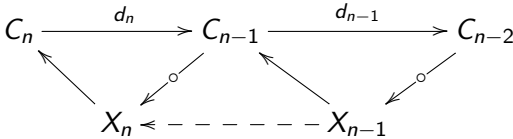
$$C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{f_n} X_n \xrightarrow{g_n} \Sigma C_n, \quad (1.1)$$

which is determined uniquely up to (non-canonical) isomorphism of triangles. Hence the following definition does not depend on auxiliary choices:

Definition

The chain complex C_\bullet is called *\mathfrak{I} -exact in degree n* if the composite map $X_n \xrightarrow{g_n} \Sigma C_n \xrightarrow{\Sigma f_{n+1}} \Sigma X_{n+1}$ belongs to \mathfrak{I} . It is called *\mathfrak{I} -exact* if it is *\mathfrak{I} -exact in degree n* for all $n \in \mathbb{Z}$.

If we recall from yesterday, in the diagram



the composition given by the stipled arrow is in \mathfrak{J} .

Lemma

Let $F: \mathfrak{T} \rightarrow \mathfrak{C}$ be a stable homological functor into some stable Abelian category \mathfrak{C} . Let C_\bullet be a chain complex over \mathfrak{T} . The complex C_\bullet is $\text{Ker}F$ exact in degree n if and only if the sequence

$$F(C_{n+1}) \xrightarrow{F(d_{n+1})} F(C_n) \xrightarrow{F(d_n)} F(C_{n-1})$$

in \mathfrak{C} is exact at $F(C_n)$.

Later we will meet homological ideals given as intersections kernels of a family of homological ideals.

Definition

An object $A \in \mathfrak{A}$ is called \mathcal{I} -projective if the functor $\mathfrak{I}(A, \square): \mathfrak{A} \rightarrow \mathfrak{Ab}$ is \mathcal{I} -exact.

We write $\mathcal{P}_{\mathcal{I}}$ for the class of \mathcal{I} -projective objects in \mathfrak{A} .

Lemma

An object $A \in \mathfrak{T}$ is \mathfrak{J} -projective if and only if $\mathfrak{J}(A, B) = 0$ for all $B \in \mathfrak{T}$.

Lemma

The class $\mathcal{P}_{\mathfrak{J}}$ of \mathfrak{J} -projective objects is closed under (de)suspensions, retracts, and possibly infinite direct sums (as far as they exist in \mathfrak{T}).

The following will supply us with projective objects.

Theorem

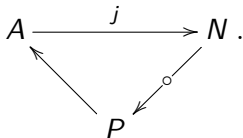
- ① *Suppose that F is \mathfrak{J} -exact and $Q \in \mathfrak{T}$ satisfies*

$$\mathfrak{T}(Q, A) = \mathfrak{C}(X, F(A))$$

for some object X in \mathfrak{C} . Then Q is \mathfrak{J} -projective.

- ② *Suppose that $\mathfrak{J} = \text{Ker } F$. Then an object P of \mathfrak{T} is projective iff $F(P)$ is projective in \mathfrak{C}*

\mathfrak{T} contains enough projectives if, for any object A in \mathfrak{T} , there exists an exact triangle of the form



with P projective and $j \in \mathfrak{J}(A, N)$. Then we can construct projective resolutions.

The phantom tower.

We will use above to construct a projective resolution.

A

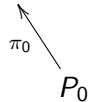
The phantom tower.

We will use above to construct a projective resolution.

$$A = N_0$$

The phantom tower.

We will use above to construct a projective resolution.

$$A = N_0$$


π_0

P_0

The phantom tower.

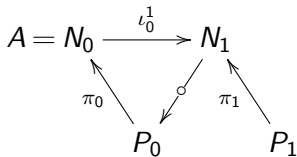
We will use above to construct a projective resolution.

$$\begin{array}{ccc} A = N_0 & \xrightarrow{l_0^1} & N_1 \\ & \swarrow \pi_0 & \searrow \circ \\ & P_0 & \end{array}$$

mapping cone of π_0

The phantom tower.

We will use above to construct a projective resolution.



The phantom tower.

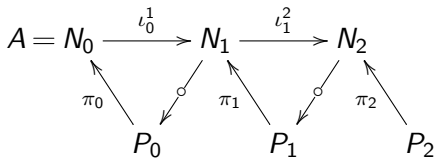
We will use above to construct a projective resolution.

$$\begin{array}{ccccc} A = N_0 & \xrightarrow{\iota_0^1} & N_1 & \xrightarrow{\iota_1^2} & N_2 \\ & \nearrow \pi_0 & \swarrow \circ & \nearrow \pi_1 & \swarrow \circ \\ & & P_0 & & P_1 \end{array}$$

mapping cone of π_1

The phantom tower.

We will use above to construct a projective resolution.



The phantom tower.

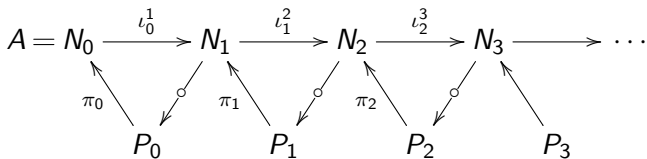
We will use above to construct a projective resolution.

$$\begin{array}{ccccccc} A = N_0 & \xrightarrow{\iota_0^1} & N_1 & \xrightarrow{\iota_1^2} & N_2 & \xrightarrow{\iota_2^3} & N_3 \\ & \nearrow \pi_0 & \searrow \circ & \nearrow \pi_1 & \searrow \circ & \nearrow \pi_2 & \searrow \circ \\ & & P_0 & & P_1 & & P_2 \end{array}$$

mapping cone of π_2

The phantom tower.

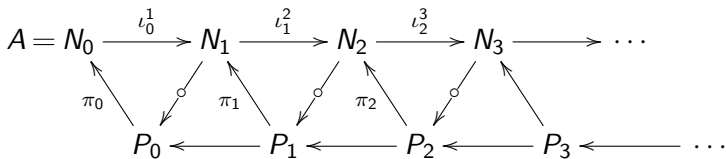
We will use above to construct a projective resolution.



etc.

The phantom tower.

We will use above to construct a projective resolution.



The successive compositions produce the projective resolution of A

Example

- 1 $\mathfrak{I} = KK^\Gamma$ for a discrete group Γ
- 2 $j \in \mathfrak{I}$ if, for all torsion subgroups $H \subset \Gamma$, $j = 0$ in KK^H
- 3 $\mathcal{P}_{\mathfrak{I}}$ coincides with the usual class of proper Γ -algebras.

Definition

Given A , its projective cover is a \mathfrak{J} -projective object P_A and $D_A \in KK^\Gamma(P_A, A)$ such that every $k \in KK^\Gamma(Q, A)$ with $Q \in \mathcal{P}_{\mathfrak{I}}$ factorizes through D .

$$\begin{array}{ccc} Q & \xrightarrow{k} & A \\ | & \nearrow D_A & \\ | & & \\ \Psi & & \\ P_A & & \end{array}$$

Example continued

Such a P_A always exists!

In fact, it is of the form $(P_{\mathbb{C}} \otimes A, D_{\mathbb{C}} \otimes 1)$, where $D_{\mathbb{C}}$ is the usual Dirac operator (mostly familiar if $B\Gamma$ is a spin-manifold)

γ element

Γ has a γ -element if $D_{\mathbb{C}}$ has the right inverse, i. e. there exists a $d \in KK^{\Gamma}(\mathbb{C}, P_{\mathbb{C}})$ (dual Dirac) such that

$$d_{\mathbb{C}} \circ D_{\mathbb{C}} = id|_{P_{\mathbb{C}}}.$$

In this case the exterior Kasparov product with $\gamma = D_{\mathbb{C}} \circ d_{\mathbb{C}} \in KK^{\Gamma}(\mathbb{C}, \mathbb{C})$ is the projection onto the complement of \mathcal{N} , the \mathfrak{J} -contractible objects, and

$$KK^{\Gamma} = \langle \mathcal{P}_{\mathfrak{J}} \rangle \oplus \mathcal{N}$$

Example continued

Given homological functor F , which vanishes on \mathfrak{J} , it has a total left derived functor $\mathbb{L}F$ with a natural transformation

$$\mathbb{L}F \rightarrow F.$$

In fact, $\mathbb{L}F(A) = F(P_A)$.

Theorem

Let $F(A) = K_*(A \rtimes_{\text{red}} \Gamma)$. Then $\mathbb{L}F(A) = K_\Gamma^*(A)$ and

$$K_\Gamma^*(A) = \mathbb{L}F(A) \rightarrow F(A) = K_*(A \rtimes_{\text{red}} \Gamma)$$

is the assembly map.

Example continued

Theorem

Suppose that Γ satisfies strong Baum-Connes conjecture, i. e. it has the γ -element equal to one. Then KK^Γ coincides with the localizing subcategory generated by $\mathcal{P}_{\mathfrak{J}}$.

Recall that f. ex. amenable groups satisfy the hypothesis. Moreover $\mathcal{P}_{\mathfrak{J}}$ is generated by homogeneous actions of Γ , hence any stable homological functor on KK^Γ which coincides with K_Γ^* on homogeneous actions is the same as K_Γ^* .

Corollary

Suppose that Γ satisfies the strong Baum-Connes conjecture, Γ acts on X and that A is a Γ - C^ -algebra in $\mathfrak{C}^* \text{alg}(X)$. If A is in $\mathcal{B}(X)$, then so is $A \rtimes_{\text{red}} \Gamma$.*

Once we have a notion of exactness for chain complexes, we can do homological algebra in the homotopy category $\text{Ho}(\mathfrak{C})$ of all chain complexes over \mathfrak{C} .

Definition

Let \mathcal{I} be a homological ideal in a triangulated category \mathcal{T} with enough projective objects. Let $F: \mathcal{T} \rightarrow \mathcal{C}$ be an additive functor with values in an Abelian category \mathcal{C} . Applying F pointwise to chain complexes, we get an induced functor $\bar{F}: \text{Ho}(\mathcal{T}) \rightarrow \text{Ho}(\mathcal{C})$. Let $P: \mathcal{T} \rightarrow \text{Ho}(\mathcal{T})$ be the functor that maps an object of \mathcal{T} to a projective resolution of \mathcal{T} . Let $H_n: \text{Ho}(\mathcal{C}) \rightarrow \mathcal{C}$ be the n th homology functor. The composite functor

$$\mathbb{L}_n F: \mathcal{T} \xrightarrow{P} \text{Ho}(\mathcal{T}) \xrightarrow{\bar{F}} \text{Ho}(\mathcal{C}) \xrightarrow{H_n} \mathcal{C}$$

for $n \in \mathbb{N}$ is called the n th *left derived functor* of F . If $F: \mathcal{T}^{\text{op}} \rightarrow \mathcal{C}$ is a contravariant additive functor, then the corresponding composite functor $\mathbb{R}^n F: \mathcal{T}^{\text{op}} \rightarrow \mathcal{C}$ is called the n th *right derived functor* of F .

More concretely, $\mathbb{L}_n F(A)$ for a covariant functor F and $A \in \mathfrak{A}$ is the homology of the chain complex

$$\cdots \rightarrow F(P_{n+1}) \xrightarrow{F(\delta_{n+1})} F(P_n) \xrightarrow{F(\delta_n)} F(P_{n-1}) \rightarrow \cdots \rightarrow F(P_0)$$

at $F(P_n)$ in degree n , where $(P_\bullet, \delta_\bullet)$ is an \mathfrak{J} projective resolution of A . Similarly, $\mathbb{R}^n F(A)$ for a contravariant functor F is the cohomology of the chain complex

$$\cdots \leftarrow F(P_{n+1}) \xleftarrow{F(\delta_{n+1})} F(P_n) \xleftarrow{F(\delta_n)} F(P_{n-1}) \leftarrow \cdots \leftarrow F(P_0)$$

at $F(P_n)$ in degree $-n$.

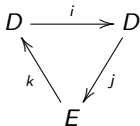
The phantom tower of A generates spectral sequences. For simplicity, we consider a homological functor $F: \mathfrak{A} \rightarrow \mathfrak{C}$. Since we are not going to use it later, we'll just describe the *exact couple* which generates it.

Let $N_n := A$ for $n \in -\mathbb{N}$ and define bigraded Abelian groups

$$D := \sum_{p,q \in \mathbb{Z}} D_{pq}, \quad D_{pq} := F_{p+q+1}(N_{p+1}),$$

$$E := \sum_{p,q \in \mathbb{Z}} E_{pq}, \quad E_{pq} := F_{p+q}(P_p),$$

and homogeneous group homomorphisms



$$i_{pq} := (\iota_{p+1}^{p+2})_* : D_{p,q} \rightarrow D_{p+1,q-1},$$

$$\deg i = (1, -1),$$

$$j_{pq} := (\varepsilon_p)_* : D_{p,q} \rightarrow E_{p,q},$$

$$\deg j = (0, 0),$$

$$k_{pq} := (\pi_p)_* : E_{p,q} \rightarrow D_{p-1,q},$$

$$\deg k = (-1, 0).$$

Since F is homological, the chain complexes

$$\cdots \rightarrow F_{m+1}(N_{n+1}) \xrightarrow{\varepsilon_{n*}} F_m(P_n) \xrightarrow{\pi_{n*}} F_m(N_n) \xrightarrow{\iota_{n*}^{n+1}} F_m(N_{n+1}) \rightarrow \cdots$$

are exact for all $m \in \mathbb{Z}$. This means that (D, E, i, j, k) is an *exact couple*.

Theorem

The ABC spectral sequence for homological functor is independent of auxiliary choices, functorial in A and abuts at $F(A)$. The second tableau involves only the derived functors:

$$E_{pq}^2 \cong \mathbb{L}_p F_q(A),$$

There is a similar result for cohomological functors.

- $\Gamma = \mathbb{Z}$
- $\mathfrak{J} = \text{Ker} : KK^{\mathbb{Z}} \rightarrow KK$

The \mathfrak{J} -projective resolution of \mathbb{C} has the form

$$\begin{array}{ccccc}
 \mathcal{K}(l^2(\mathbb{Z})) & \xrightarrow{\quad} & \mathbb{C} \simeq \Sigma c_0(\mathbb{Z}) & \xrightarrow{\quad} & 0 \\
 & \swarrow \pi & \searrow \circ & \swarrow \Sigma & \searrow \\
 & & c_0(\mathbb{Z}) & \xleftarrow{1-\sigma} & c_0(\mathbb{Z})
 \end{array}$$

The projective cover of \mathbb{C} is just the mapping cone

$$c_0(\mathbb{Z}) \rightarrow c_0(\mathbb{Z}) \rightarrow \Sigma C_{1-\sigma}.$$

But this is just the rotated exact triangle associated to the extension

$$0 \rightarrow \Sigma c_0(\mathbb{Z}) \rightarrow C_0(\mathbb{R}) \rightarrow c_0(\mathbb{Z}) \rightarrow 0,$$

the $*$ -homomorphism $C_0(\mathbb{R}) \rightarrow c_0(\mathbb{Z})$ given by the evaluation $f \rightarrow f|_{\mathbb{Z}}$.

Conclusion

$P_{\mathbb{C}} = C_0(\mathbb{R}^2)$, $D = \bar{\partial}$, the usual Dirac operator (or rather its phase),

$$K_{\mathbb{Z}}^*(A) = K_*((A \otimes C_0(\mathbb{R}^2)) \rtimes \mathbb{Z}) \rightarrow K_*(A \rtimes \mathbb{Z}),$$

where the assembly map is given by the product with Dirac operator.

The spectral sequence computing $K_{\mathbb{Z}}^*(A)$ becomes the six term exact sequence in K-theory associated to the extension

$$\Sigma(A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z} \rightarrow (A \otimes C_0(\mathbb{R}^2)) \rtimes \mathbb{Z} \rightarrow (A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z}$$

Since $(A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z} \simeq A \otimes \mathcal{K}$, this is just the usual Pimsner-Voiculescu exact sequence.

A discrete quantum group \mathbb{G} is a unital C^* -algebra $C(\mathbb{G})$ with a compatible coalgebra structure

$$\Delta : C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes_{\min} C(\mathbb{G}).$$

satisfying the non-degeneracy conditions

$$\Delta(C(\mathbb{G})(1 \otimes C(\mathbb{G}))) = C(\mathbb{G}) \otimes_{\min} C(\mathbb{G}) = (C(\mathbb{G}) \otimes 1) \Delta(C(\mathbb{G})),$$

The additional structures are antipode and Haar state μ . There is an associated dual *compact* quantum group $(C_0(\widehat{\mathbb{G}}), \widehat{\Delta})$ which carries its own Haar weight. $C_0(\widehat{\mathbb{G}})$ is nonunital (or finite dimensional) and the coproduct has the form

$$C_0(\widehat{\mathbb{G}}) \rightarrow \text{Mult}(C_0(\widehat{\mathbb{G}}) \otimes_{\min} C_0(\widehat{\mathbb{G}}))$$

An action of \mathbb{G} on a C^* -algebra A is the same as a coaction of $\widehat{\mathbb{G}}$ on A , i.e. a $*$ -homomorphism $\rho : A \rightarrow A \otimes_{\min} C(\widehat{\mathbb{G}})$ satisfying the condition

$$\iota \otimes \Delta \circ \rho = \rho \otimes \iota \circ \rho : A \rightarrow A \otimes_{\min} C(\widehat{\mathbb{G}}) \otimes_{\min} C(\widehat{\mathbb{G}}).$$

and nondegeneracy $\rho(A) \left(1 \otimes C(\widehat{\mathbb{G}}) \right) = A \otimes_{\min} C(\widehat{\mathbb{G}})$.

There is a notion of the category of \mathbb{G} - C^* -algebras and corresponding notion of the bivariant $KK^{\mathbb{G}}$ functor satisfying all the expected properties. In particular we get the corresponding triangulated $KK^{\mathbb{G}}$ category.

Examples

- A discrete group Γ . $C(\mathbb{G}) = C_{red}^*(\Gamma)$, the coproduct is given by $\gamma \mapsto \gamma \otimes \gamma$ and $C_0(\widehat{\Gamma}) = c_0(\Gamma)$.
- A compact group G . $C(\mathbb{G}) = C(G)$, the coproduct is given by $\Delta(f)(g, h) = f(gh)$ and $C_0(\widehat{\Gamma}) = C^*(G)$.

We assume that G is a compact, connected group.

- $\mathfrak{T} = \text{KK}^{\hat{G}}$, the KK-category of G -coalgebras,
- $\mathfrak{J} = \text{Ker} : \text{KK}^{\hat{G}} \rightarrow \text{KK}$
- $F(A) = K^*(A \rtimes \hat{G})$

Theorem

Baum-Connes for coactions

$$\langle \mathcal{P}_{\mathfrak{J}} \rangle = \text{KK}^{\hat{G}}$$

The spectral sequence computing $K_{\hat{G}}^*(A) = \mathbb{L}F(A)$ becomes a spectral sequence for the K-theory of $B = A \rtimes \hat{G}$:

$$E_{p,q}^* \implies K_{p+q}(A \rtimes \hat{G})$$

and the E^2 -term has form

$$E_{-p,q}^2 = H^p(R_G, K_q(A))$$

In terms of a G -algebra $B = A \rtimes \hat{G}$ we get

Corollary

Let G be a connected compact group and B a separable G - C^ algebra. Then*

$$K_*^G(B) = K_*(B \rtimes G) = 0 \implies K_*(B) = 0.$$

Let $\widehat{\mathbb{G}} = SU_q(2)$. In this particular case \mathbb{G} is "torsion free" and the BC says that the trivial \mathbb{G} -module \mathbb{C} is contained in the localising category generated by $C(\mathbb{G})$. More generally the following stronger statement holds.

Theorem

Let $\mathbb{G} = \widehat{SU_q(2)}$. The $KK^{\mathbb{G}}$ -category coincides with the subcategory generated by C^* -algebras with trivial \mathbb{G} actions

Note that, in distinction to the group case, the $KK^{\mathbb{G}}$ -category has no tensor product, so the Dirac - dual Dirac method does not work without major modification.